Quantum Time-Frequency Transforms

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Abstract

Time-frequency transforms represent a signal as a mixture of its time domain representation and its frequency domain representation. We present efficient algorithms for the quantum Zak transform and quantum Weyl-Heisenberg transform.

1 Introduction

The Fourier transform is an operator that expresses a time-dependent signal as a sum (or integral) of periodic signals. In other words the Fourier transform changes a function of time $s(t)$ into a function of frequency $S(\omega)$. If a signal is a function of time it said to be in the "time domain" and if it is a function of frequency it is said to be in the "frequency domain". For signals whose spectrum is changing in time, i.e. nonstationary signals, sometimes the best description is a mixture of the time and frequency components. Signal representations which mix the time and frequency domains are called, naturally enough, "time-frequency representations" and are often used to describe time-varying signals for which the pure frequency or Fourier representation is inadequate [2],[3]. A familiar example of a time-frequency representation is a musical score, which describes *when* (time) certain *notes* (frequency) are to be played.

Formally speaking for our present purposes, a quantum signal is simply a quantum state $|\psi\rangle$ where the Hilbert space is the group algebra $\mathbb{C}[G]$ of a finite abelian group G. The Quantum Fourier Transform (QFT) is central to the important quantum algorithms for factoring and discrete logarithm. Mathematically speaking, the Quantum Fourier Transform is a linear operator on the Hilbert Space $\mathbb{C}[G]$ which is a change of basis from

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the basis of group elements $\{|g_1\rangle, ..., |g_{|G|}\rangle\}$ to the basis of characters of G, $\{|\chi_1\rangle, |\chi_2\rangle, ..., |\chi_{|G|}\rangle\}$.

We present efficient algorithms for quantum versions of the Zak and Weyl-Heisenberg transforms. Both these time-frequency tranforms can be seen as generalizations of Fourier transforms and the quantum algorithms make heavy use of the Quantum Fourier Transform. We follow the theory and notation of [4] and recommend this book as background to this material.

2 Zak Transforms

2.1 Background

Let A be a finite, abelian group, A^* the group of characters of A (note: in this paper * does *not* mean conjugation), $B \leq A$ a subgroup of A, $B_* = \{a^* \in A^* : a^*(b) = 1, b \in B\}$ the dual to B, and $f \in C[A]$, the group algebra of A. Define

$$Z(B)f \in C[A \times A^*]$$

by the formula

$$Z(B)f(a, a^*) = \sum_{b \in B} f(a + b)\overline{a^*(b)}.$$

$F = Z(B)f$ is called the Zak transform of f over B. A simple calculation shows that $F(a + b, a^* + b_*) = a^*(b)F(a, a^*)$ where $b \in B$ and $b_* \in B_*$. Therefore F is determined by its values on a set of coset representatives of $B \times B_*$ in $A \times A^*$ and thus conceptually we may think of F as a function on T where T is a set of coset representatives. Since

$$\frac{|A \times A^*|}{|B \times B_*|} = \frac{|A|^2}{|B||B_*|} = |A|$$

we have the same number of degrees of freedom with which we started. Notice that if B contains only the identity, i.e. is the trivial subgroup, then $Z(B)f(a, a^*) = f(a)$ and is basically the identity map. Also notice that if $B = A$ then $Z(A)f(0, a^*) = \langle a^*|f\rangle$ and therefore $Z(A)f$ is basically the Fourier transform of f. So the Zak transform mediates between the time domain and frequency domain depending on the subgroup B.

Consider the function $f(a_0) = \delta(x - a_0)$ which is 1 on a_0 and 0 otherwise. Applying the above formula for the Zak transform yields $F(a, a^*) = a^*(a - a_0)$ for $a \in a_0 + B, a^* \in A^*$ and 0 otherwise. But since F is determined by its values on a set of coset representatives of $B \times B_*$ in

$A \times A^*$ let us introduce such a set of representatives $T = T_1 \times T_2 = \{(x_i, a_j^*)\}$ where $T_1 = \{x_i\}$ is a set of coset representatives of B in A and $T_2 = \{a_j^*\}$ is a set of coset representatives of B_* in A^*. Bearing in mind the above transformation of a delta function, we now offer our definition of the Quantum Zak Transform (QZT) (with respect to T) by

$$|a\rangle \mapsto \frac{1}{\sqrt{|B|}} \sum_{a_j^* \in T_2} a_j^*(x_a - a)|x_a\rangle|a_j^*\rangle.$$

where $x_a \in T_1$ is the coset representative of a. Now notice that $x_a - a \in B$. Therefore a_j^* is restricted to B and therefore can be considered to be a character of B, i.e. an element of B^*, and this restriction is independent of the choice of coset representative, i.e. it is *natural* or *canonical*. Therefore an equivalent formulation of the QZT is given by

$$|a\rangle \mapsto \frac{1}{\sqrt{|B|}} \sum_{b^* \in B^*} b^*(x_a - a)|x_a\rangle|b^*\rangle.$$

The only difference in these two formulations is in the *interpretation* of the observed content of the second register.

2.2 The Quantum Algorithm

We now show that the QZT is efficiently implementable. Define $P(B)$ to be the transform

$$P(B)|a\rangle = |x_a\rangle|x_a - a\rangle$$

which decomposes a into its coset representative and the corresponding element of B. P is clearly unitary and efficiently implementable. After applying $P(B)$ we apply the Quantum Fourier Transform (over the group B, denoted F_B) to the second register. This results in the state

$$\frac{1}{\sqrt{|B|}} \sum_{b^* \in B^*} b^*(x_a - a)|x_a\rangle|b^*\rangle.$$

Therefore the QZT is simply $Z(B) = (I \otimes F_B) \circ P(B)$.

3 Weyl-Heisenberg Transforms

3.1 Background

Define $g_{(x,x^*)}(a) = g(a - x)x^*(a)$ to be the *time-frequency* translate of g by (x, x^*) where $g \in C[A]$. We will use time-frequency translates to form

orthonormal bases so we also require $|g| = 1$. Let $\Delta = B \times B_*$ and $(g, \Delta) = \{g_{(x,x^*)} : (b, b_*) \in \Delta\}$. We call (g, Δ) a *W-H system over* Δ *with window* g. A basic result ([4], Theorem 12.1 *corrected* version) is that (g, Δ) is an orthonormal basis of $C[A]$ if and only if for all $(a, a^*) \in A \times A^*$ we have $|G(a, a^*)| = \sqrt{\frac{|B|}{|A|}}$ where the Zak tranform is taken over B. Because von Neumann measurements must be unitary we will restrict our attention to window functions g which satisfy this constraint. Utilizing POVMs one could consider implementing nonorthonormal W-H systems but we will not address this in this note. This orthogonality constraint together with the earlier observation that G is determined by its values on a set of coset representatives of $B \times B_*$ in $A \times A^*$ implies that orthonormal W-H systems are in bijective correspondence with the set of all $|A|$-tuples of complex numbers with modulus $\sqrt{\frac{|B|}{|A|}}$. In this note we will restrict the W-H systems under consideration by assuming that for each $(a, a^*) \in A \times A^*$ the phase of $G(a, a^*)$ is a rational fraction of 2π which we can compute in polynomial time. Whether or not this last assumption is excessively restrictive would depend on the intended application. Notice that if g is the constant function $g = \frac{1}{\sqrt{|A|}}$ and $\Delta = \{0\} \times A^*$ then (g, Δ) is the (normalized) Fourier basis, $G(a, a^*) = \frac{1}{\sqrt{|A|}}$ and this restriction holds trivially.

We define the Quantum Weyl-Heisenberg Transform (QWHT) by

$$|\psi\rangle \mapsto \sum_{(b,b_*)\in\Delta} \langle\psi|g_{(b,b_*)}\rangle|b, b_*\rangle.$$

In other words, the QWHT expresses $|\psi\rangle$ in the orthonormal basis of time-frequency translates of the window function.

3.2 The Quantum Algorithm

Let

$$f = \sum_{(b,b_*)\in\Delta} \alpha(b, b_*)g_{(b,b_*)}$$

i.e. the α's are the coeffients of the WH-expansion of f. Define

$$P(a, a^*) = \sum_{(b,b_*)\in\Delta} \alpha(b, b_*)b_*(a)\overline{a^*(b)}.$$

Notice that P is Δ-periodic and that the α's are, by definition, the Fourier coefficents (over $A \times A^*$) of P. A fundamental result ([4], Theorem 7.5)

states that $F = GP$. This result suggests an algorithm for computing the WH-coefficients of f, namely compute the Fourier coefficients of $P = \frac{F}{G}$.

Define $\Phi(g)$ to be the unitary transformation which acts on the Hilbert space $C[T]$ (recall T is the set of coset representatives of $B \times B_*$ in $A \times A^*$) by

$$|x_i\rangle|a_j^*\rangle \mapsto \frac{1}{G(x_i, a_j^*)}|x_i\rangle|a_j^*\rangle.$$

Since the phase of $G(x_i, a_j^*)$ is, by assumption, a rational fraction of 2π computable in polynomial time we may efficiently implement $\Phi(g)$ by the phase kickback technique described in [1]. Finally in order to complete our description of the algorithm, we must assume that we are given an explicit isomorphism between A and A^*. These groups are isomorphic, though not canonically so. Therefore in any computational situation we provide an explicit isomorphism by choosing an explicit computational representation of the groups A and A^*. This isomorphism induces explicit isomorphisms between B and B^* and between the factor group A/B and B_*. We will see shortly how we will employ these three interrelated isomorphisms. We will highlight this interrelation, and abuse notation, by using the symbol ϕ to refer to all three of these isomorphisms, allowing for context to make the usage clear. As in the case of the Zak transformtion, these isomorphisms are simply *reinterpretations* of the contents of the registers.

Our QWHT is the sequence $F_{B_* \times B} \circ \Phi(g) \circ Z(B)$. Let us see how this unitary transformation acts on $|a\rangle$. We have

$$Z(B)|a\rangle = \frac{1}{\sqrt{|B|}} \sum_{a_j^* \in T_2} a_j^*(x_a - a)|x_a\rangle|a_j^*\rangle$$

and then after applying $\Phi(g)$ we obtain:

$$\frac{1}{\sqrt{|B|}} \sum_{a_j^* \in T_2} \frac{a_j^*(x_a - a)}{G(x_a, a_j^*)}|x_a\rangle|a_j^*\rangle$$

which by the fundamental result discussed above equals:

$$\frac{1}{\sqrt{|B|}} \sum_{b^*} P(x_a, b^*)|x_a\rangle|b^*\rangle$$

where we are now considering the contents of the second register to be an element of B^*. We now utilize our explicit isomorphisms to reinterpret the

contents of the first register as an element of B_* and the contents of the second register as an element of B:

$$\frac{1}{\sqrt{|B|}}\sum_b P(b_*,b)|b_*\rangle|b\rangle = \frac{1}{\sqrt{|B|}}\sum_{\phi(a_j^*)} P(\phi(x_a),\phi(b^*))|\phi(x_a)\rangle|\phi(b^*)\rangle.$$

By applying the final transformation in the sequence $F_{B_*\times B}$ we obtain our desired expansion:

$$\sum_{(b,b_*)\in\Delta}\langle a|g_{(b,b_*)}\rangle|b\rangle|b_*\rangle.$$

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